Generalized Tetrahedrons and Sculptural Variations

Nathaniel Friedman
Professor Emeritus Mathematics, SUNY-Albany
nat.isama77@gmail.com

Robert Krawczyk
Professor Architecture, Illinois Institute of Technology, Chicago
krawczyk@iit.edu

Introduction

A basic tetrahedron is shown in Figure 1(a) consisting of four equilateral triangles. The number of vertices V = 4, the number of edges E = 6, and the number of faces F = 4. Thus Euler's formula yields V – E + F = 4 – 6 +4 = 2 corresponding to a sphere.

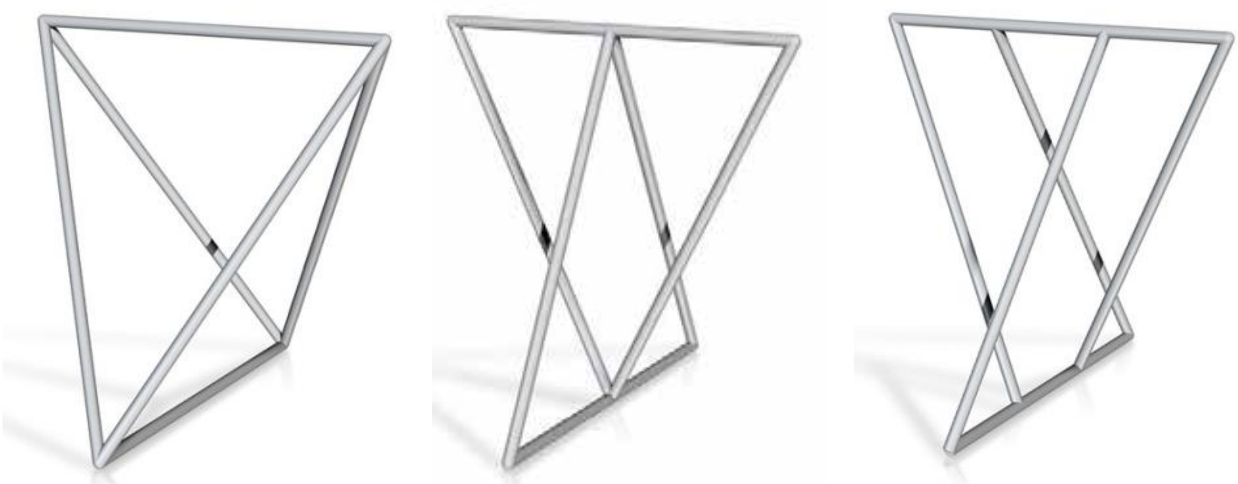


Figure 1. (a) Basic Tetrahedron. (b) Generalization 1. (c) Generalization 2.

First Stage of Generalization

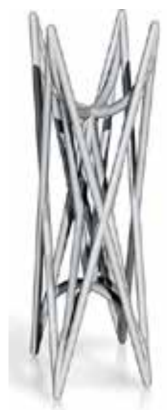
We consider the four points (vertices) in Figure 1 as one pair corresponding to the endpoints of the upper edge and one pair corresponding to the endpoints of the opposite lower edge. The first generalization is to connect, by a straight line, each endpoint of these two edges to an arbitrary point on the opposite edge. For example, if both endpoints are connected to the midpoint of the opposite edge, then we obtain the generalized tetrahedron in Figure 1(b). First note that there are six vertices, three each on the upper and lower edges. There are 8 edges and four identical faces and each face is bounded by four edges. We have V – E + F = 6 – 8 + 4 = 2 as above for a basic tetrahedron. The four faces are not planar. However, if a wire frame in the shape of a face was dipped in a soap solution, the resulting soap film minimal surface with the wire as boundary would be a hyperbolic surface. Thus we can consider Generalization 1 corresponding to a tetrahedron with four bounding hyperbolic surfaces.

For the next example, connect each endpoint to the corresponding point 1/3 of the distance between points on the opposite edge. The resulting tetrahedron is shown in Figure 1(c). In this case there are four hyperbolic faces (surfaces), each bounded by five edges since there is an extra vertex in the middle of two edges on the same line. In this case the number of vertices is 8 and the number of edges is 10. Thus V –E + F = 8 – 10 +4 = 2 as above.

Second Stage of Generalization

The second stage of generalizing a tetrahedron is to replace some or all straight edges by curves. If we replace the upper and lower edges by curves in Figure 1 (a), we obtain Figure 2.



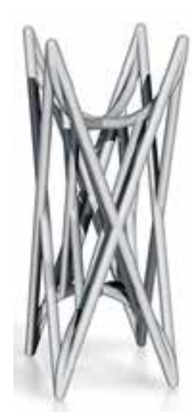


Figure 2. Figure 3. Figure 4. Figure 5. Figure 6. Figure 7.

Figure 2-7. Curve-linear tetrahedron 1 with four conical faces and five variations.

Note that there are four faces where each face has two straight edges and one curved edge. If we dipped a wire model of a face in a soap solution, there would be a soap film minimal surface that would be conical shaped. We can rotate a copy of Figure 2 a quarter turn and combine it with Figure 2 to obtain the variation in Figure 3. Note that Variation 1 is stable since it sits on four points. Variation 1 suggests an architectural application as a supporting tower. One way to add strength in Figure 2 is to apply Generalization 1 to Figure 2 by adding straight edges from the ends of the curves to the center point of the opposite curve, as in Figure 4. Lastly, a quarter turn rotation of Variation 2 can be combined with Variation 2 to obtain Variation 3 in Figure 5. This is Variation 1 with triangulation for added strength. Another way to add strength in Figure 2 is to insert a tetrahedron inside by connecting the points that divide the curved ends into thirds, as Figure 6. Variation 4 can be rotated a quarter turn and combined with itself as in Figure7.

Groups

A larger support can be constructed by so-called grouping. A group corresponding to Figure 2 is shown in Figure 8. A group corresponding to Variation 2 in Figure 1 is shown in Figure 9.

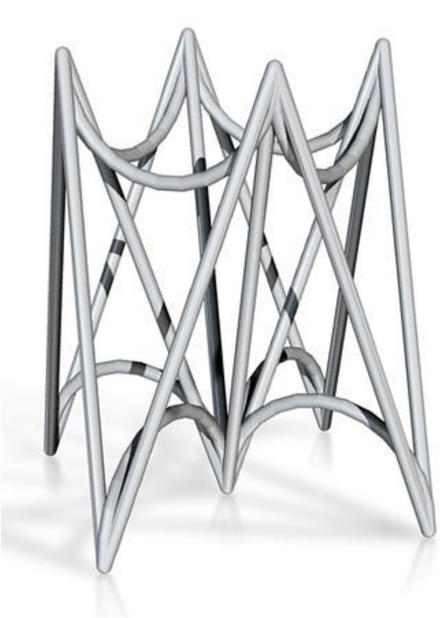


Figure 8. Group 1.

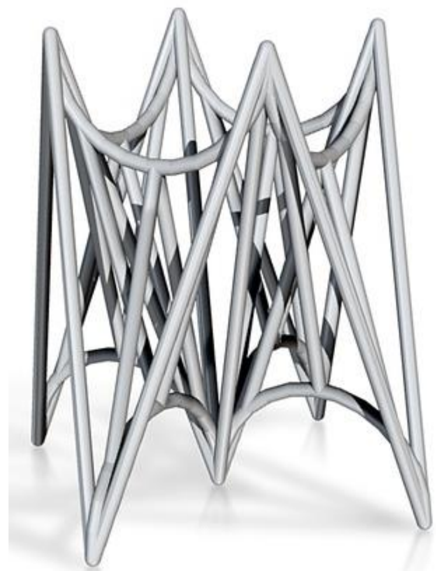
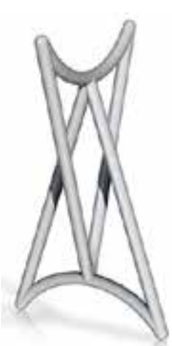
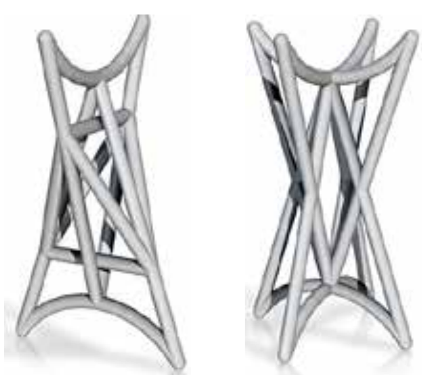


Figure 9. Group 2.

Generalization 1.

We will now replace the upper and lower edges in Generalization 1 in Figure 1(b) by curves as in Figure 12. In this case there are six vertices, eight edges, and four hyperbolic faces, each bounded by four edges. Thus V – E + F = 6 – 8 +4 = 2. By connrecting 1/3 points, an inner tetrahedron is formed as in Figure 13(a). A quarter turn rotation of the tetrahedron in Figure 10 can be combined with itself, as in Figure 11(b). Four copies of Figure 12 and four copies of variation 7 can be grouped as in Figure 12 (a) and (b).





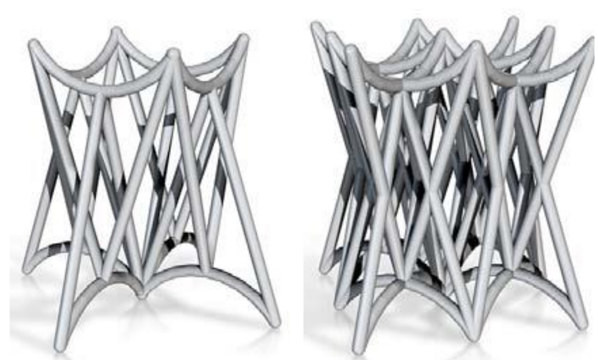


Figure 10. Figure 11. (a-b) Variations 6 and 7. Figure 12. (a-b) Groups 3 and 4.
Figure 10-12. Curve-linear tetrahedron 2 with four hyperbolic faces; its variations and groups.

Generalization 2.

We will now consider Generalization 2 in Figure 1(c) combined with a quarter turn rotation as in Figure 13. This variation could serve as a table support.

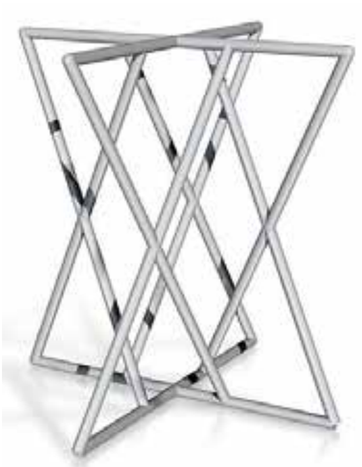
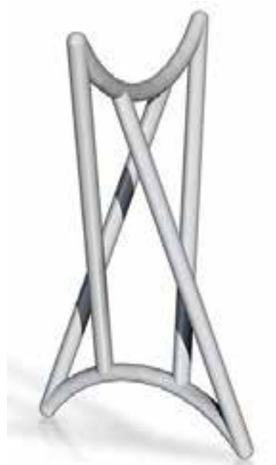


Figure 13. Variation 8. Table support.



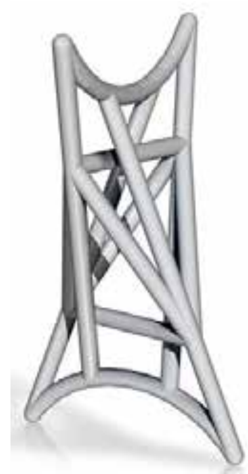


Figure 14. (a) Curve-linear Tetrahedon 3 with four hyperbolic faces. (b) Variation 9.

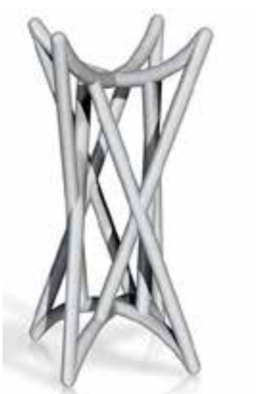


Figure 15. (a) Variation 10.

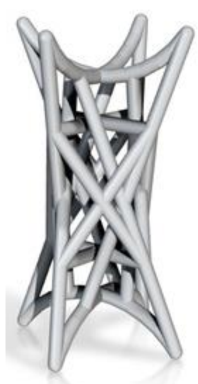


Figure 15. (b) Variation 11.

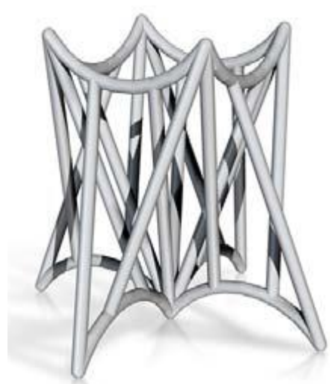


Figure 16. (a) Group 4.

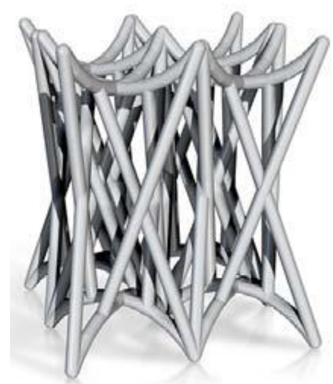


Figure 16. (b) Group 5.

Next replace the upper and lower edges in Figure 1(c) by curves as in Figure 16(a). In this case there are V = 8 vertices, E = 10 edges, and F = 4 hyperbolic faces; hence V – E + F = 8 – 10 + 4 = 2. An inner tetrahedron can be formed by connecting the one-third points as in Figure 14 (b). The tetrahedron in Figure 14(a) can be combined with a quarter-turn rotation as in Figure 15(a). Variation 6 can be combined with a quarter-turn rotation as in (b). Groups corresponding to Figure 14(a) and variation 10 are shown in Figures 16(a) and (b) respectively.

Replacing all edges with curves

We shall now replace all the straight edges in Figure 1(b) with curves. We are going to replace the upper and lower edges with half circles and replace the other four edges with quarter circles. It is convenient to begin with Figure 17(a). For the circle from left to right, choose the lower half circle and the upper two concave quarter circles. For the other circle, front to back, choose the upper half circle and lower concave quarter circles. The resulting shape is shown in Figure 17(b). This shape is referred to as a *femisphere* and was discovered by the British woodworker J.Roberts many years ago. Here *femi* refers to the curved edges (Google, femisphere). A femisphere made of wire is a toy that rolls in a wobbly way. The half circles will leave tracks in sand that are spaced arcs. Two examples of solid wooden femispheres are shown in Figure 18. These were turned by Clyde Collier of the Gulf Coast Wood Turners Association, Houston, Texas.

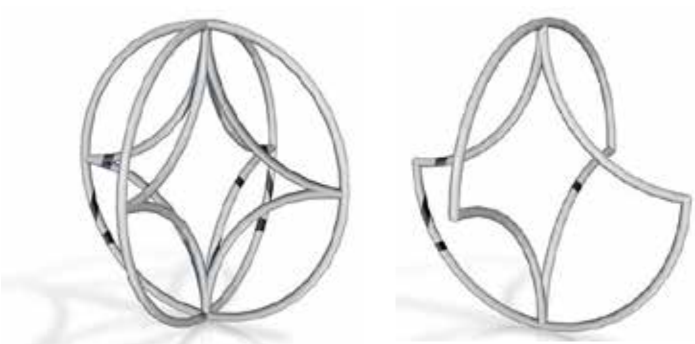


Figure 17. Femisphere as curved tetrahedron 1 with four hyperbolic faces. (a) Two femipheres (b) one femisphere

Figure 18. Clyde Collier, wooden femispheres.

Charles Perry (1929-2011), an eminent sculptor whose work was inspired by mathematics, independently discovered the femisphere by starting with a sphere that is then transformed by inverting the arcs of the surface on the lines of the stitching of a baseball. Perry referred to this shape as a *mace* and positioned a mace in a horizontal position as in Figure 18. Three Mace sculptures by Perry are shown in Figure 19. We note that both Roberts and Perry discovered the shape by considering a sphere rather than a tetrahedron.

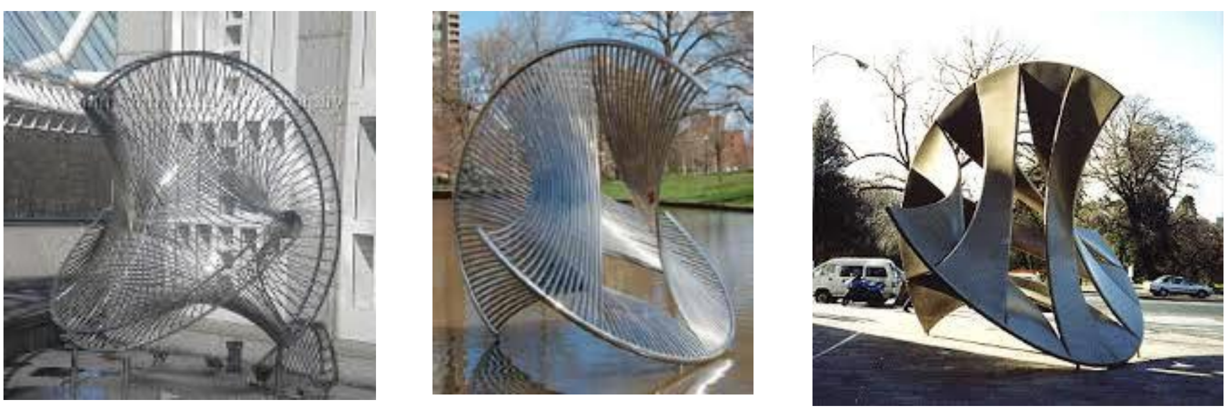


Figure 19. (a) Early Mace, Peachtree Center, Atlanta, GA., Steel, 12 ft, 1971. (b) Harmony, Bushnell Park, Hartford, CT., Stainless Steel, 12 ft, 1990. (c) Shell Mace, Shell Oil Co. Ltd., Melbourne, Australia, Aluminum, 28 ft., 1989.

Variations.

A quarter turn rotation of the femisphere is joined with itself in Figure 20. The center points of the quarter circles in Figure 17(b) can be joined by straight lines to form an inner linear tetrahedron, as in Figure 21. The outer femisphere can be considered as a curved tetrahedron. In general, the Mother and Child is a historical subject in sculpture, where the outer form protects the inner form. With this in mind, we can view the sculpture as a Curved Mother and Linear Child.

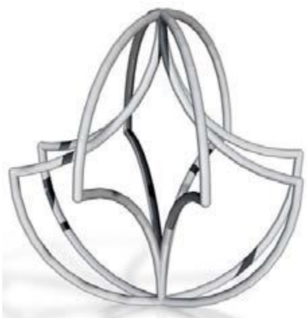


Figure 20. Variation 12.

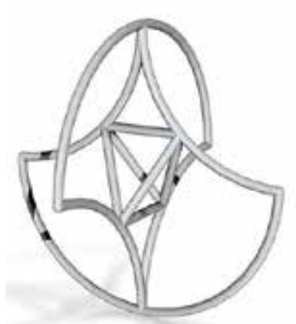


Figure 21. Variation 13.

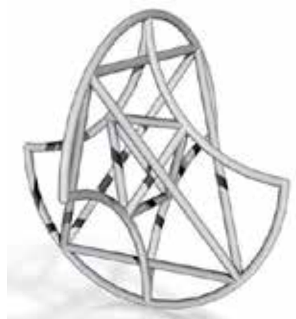


Figure 22. Variation 14.

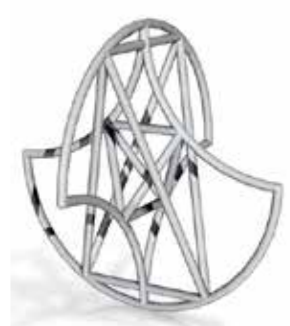


Figure 23. Variation 15.

We shall now consider some further embellishments of Figure 17(b). The midpoints of the convex half circles in (b) can be connected by straight lines to obtain a second inner linear tetrahedron, as in Figure 22.

If the points one-third of the way along the convex quarter circles in Figure 9 are connected by straight lines, then a second taller inner linear tetrahedron is obtained as in Figure 23.

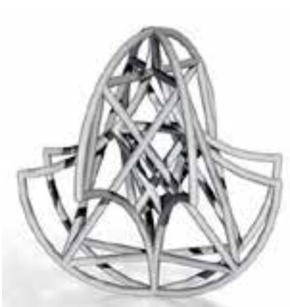


Figure 24. Variation 16.

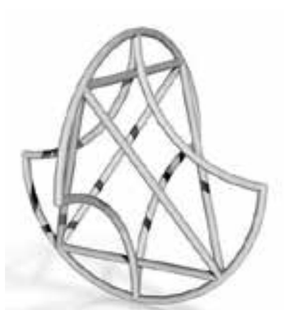


Figure 25. Variation 17.

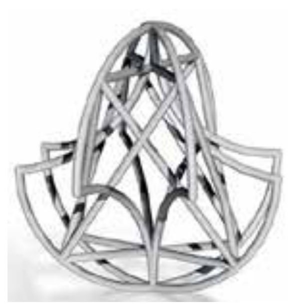


Figure 26. Variation 18.

Figure 27. Variation 19.

If the structure in Figure 22 is rotated a quarter turn and joined with the original structure, than the form in Figure 24 is obtained.

If we only connect the midpoints of the convex quarter circles in Figure 17 (b), than we obtain Figure 22 without the inner tetrahedron, as in Figure 25. Variation 17 is now rotated a quarter turn and combined with itself to obtain Variation 18 in Figure 26. For additional variations, we can embellish Figure 17(a) as shown in Figures 27 – 29.

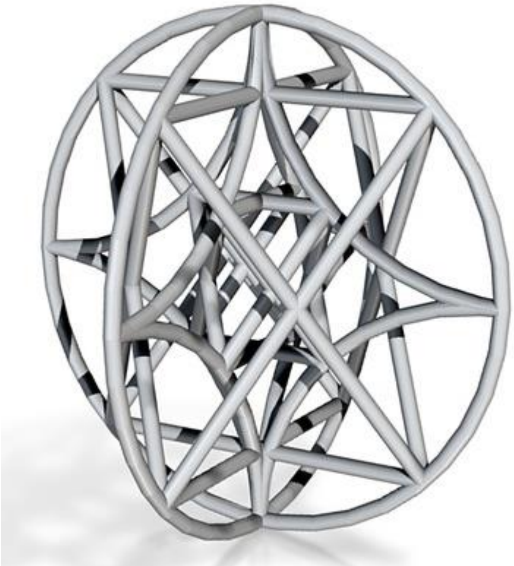


Figure 28. Variation 20.

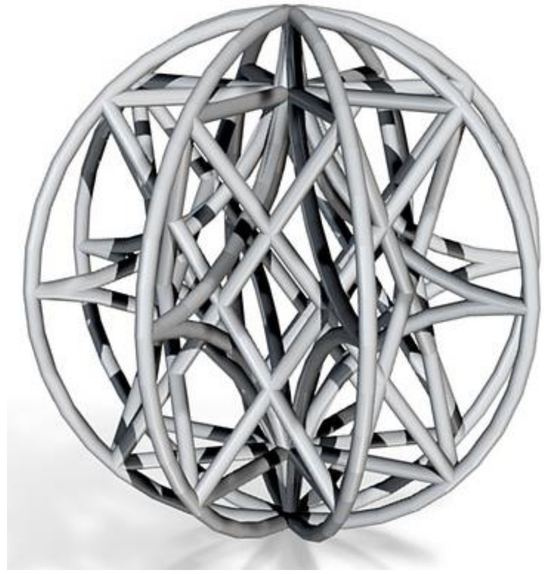


Figure 29. Variation 21.

Additional Versions of Figure 1(a).

We will now consider further modifications of the basic tetrahedron Figure 1(a). First replace the diagonal edges by convex curves, as in Figure 30(a). Join a quarter turn rotation of this figure with itself, as in (b). A corresponding group is shown in (c).

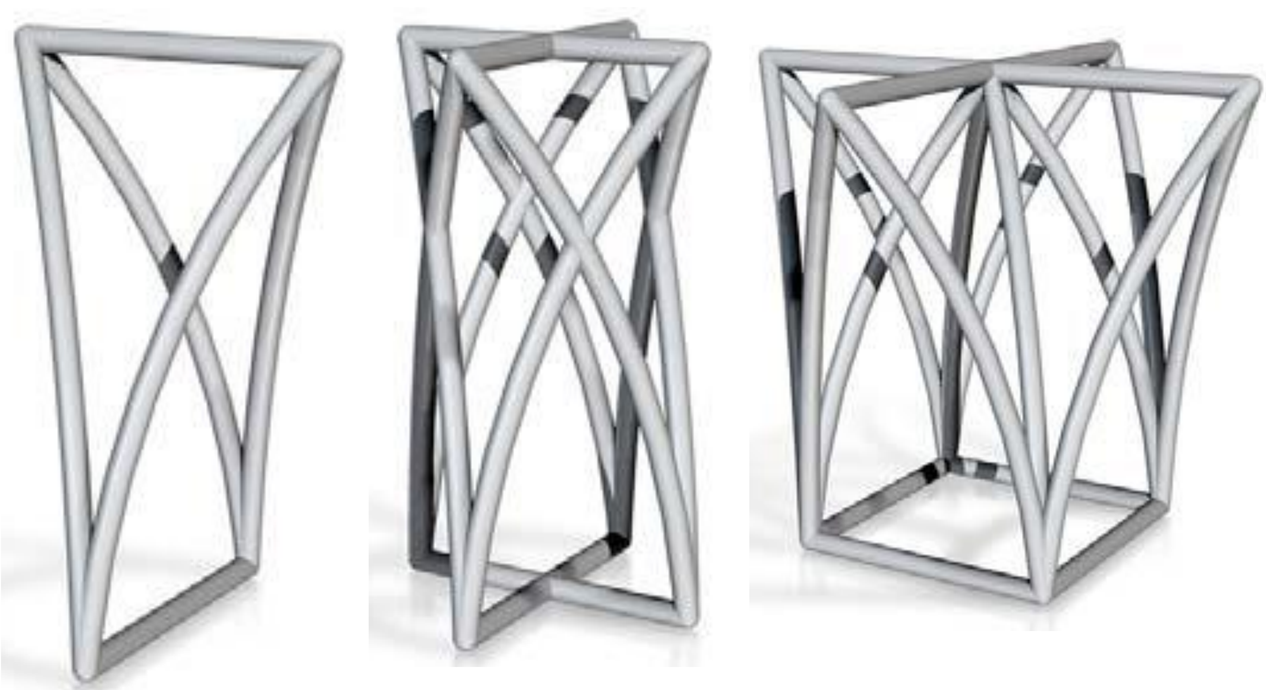


Figure 30. (a) Curve-linear Tetrahedron 5 with four cylindrical faces. (b) Quarter turn rotation of (a) joined with(a). (c) Group corresponding to (a).

The next modification is to replace the upper and lower edges in Figure 30(a) by concave curves as in Figure 31(a). A quarter turn joining and a group are shown in (b) and (c) respectivly.

Next all edges in Figure 1(b) are replaced by concave curves, as in Figure 32(a). A quarter turn of (a) is joined to (a) in (b) and a group is shown in (c).

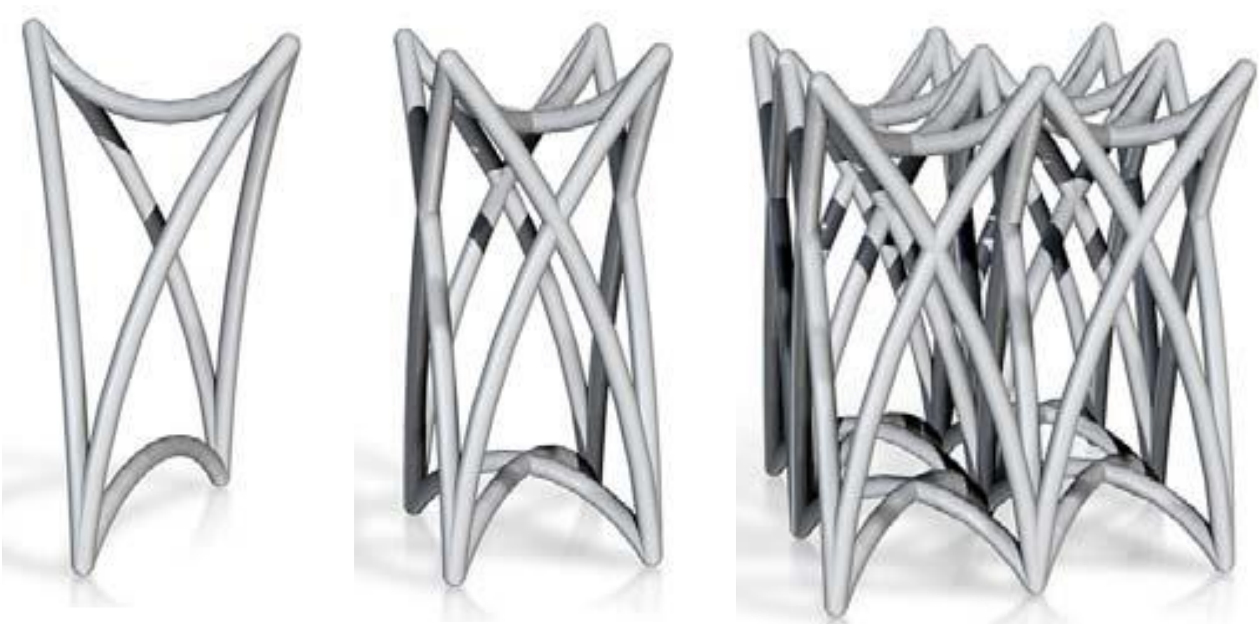


Figure 31. (a) Curved Tetrahedron 3 with four hyperbolic faces. (b) Quarter turn rotation of (a) joined with (a). (c) Group corresponding to (a).

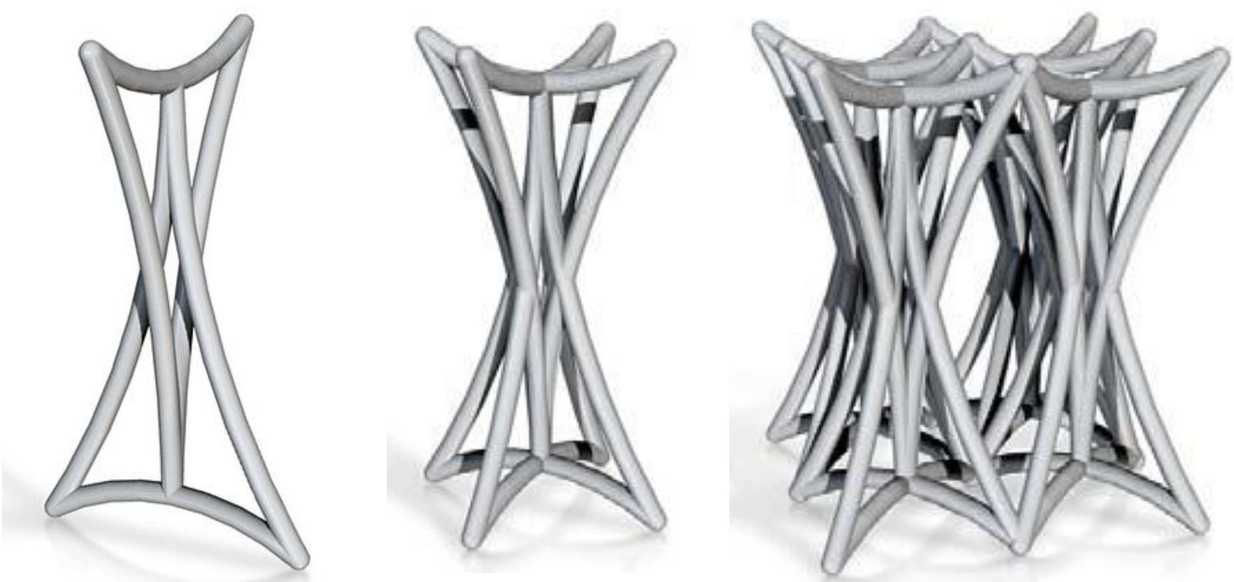


Figure 32. (a) Curved Tetraheron 6 with four hyperbolic faces. (b) Quarter turn rotation of (a) joined with (a). (c) Group corresponding to (a).